

Bilevel Model Predictive Control[★]

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Abstract: Distributed model predictive control (MPC) can be categorized based on information flows and the order of computation between the distributed controllers. Hierarchical MPC has a supervisory MPC layer that interacts with a low-level MPC layer, and both layers of MPC are engineered to ensure stability. Noncooperative MPC has two MPC controllers with different (often competing) objective functions, and the two controllers compute their control input simultaneously. This paper defines and studies bilevel MPC, which is a distributed MPC with a Stackelberg game structure. In our setting, there are two MPC controllers: The bilevel MPC chooses a sequence of control inputs, and communicates this to a lower-level MPC controller that chooses its own inputs based on the inputs of the bilevel MPC. The objective of the bilevel MPC is to select an optimal sequence of inputs knowing the behavior of the lower-level MPC controller. We first formulate the bilevel MPC, and then give counterexamples that demonstrate how the interconnections in bilevel MPC can lead to complex behaviors like loss/gain of controllability or loss/gain of stability. Next, we provide sufficient conditions under specific dynamics and cost functions for stabilizability of bilevel MPC and stability of a given bilevel MPC controller. An approach to synthesize a stable bilevel MPC controller is also described. We conclude with one example to demonstrate a duality-based algorithm for solving the optimization problem for bilevel MPC, and another example to demonstrate an integer programming algorithm for solving the optimization problem for bilevel MPC of an electric utility changing electricity prices to perform demand response of a home's air conditioner controlled by linear MPC.

Keywords: Nonlinear predictive control, dynamic games, parametric optimization

1. INTRODUCTION

Distributed model predictive control (MPC) schemes (Camponogara et al., 2002; Venkat et al., 2005; Rawlings and Mayne, 2009; Scattolini, 2009; Raimondo et al., 2009; Ma et al., 2011; Farina and Scattolini, 2011; Ferramosca et al., 2013) can be categorized based on the information flows and the order of computation between the distributed controllers. The aspects that can vary include: the number of MPC controllers, whether the controllers have cooperative or noncooperative objectives, if state information is communicated, if control information is communicated, and whether there are any additional coordination signals transmitted between controllers.

Hierarchical MPC (Scattolini and Colaneri, 2007; Picasso et al., 2010; Vermillion et al., 2014) considers a setting with a supervisory MPC layer that interacts with a low-level MPC layer, and state and control information is passed between the two layers. The low-level MPC is responsible for stabilization through control at fast time-scales, and the supervisory MPC is responsible for high-level control

(e.g., implementing logic or changing set points). In hierarchical MPC, both levels of MPC are engineered to ensure closed-loop stability after the interconnection.

A related type of distributed MPC is noncooperative MPC (Venkat et al., 2005; Rawlings and Mayne, 2009). Here, two coequal MPC controllers communicate control information between the two controllers. However, the two controllers have differing objective functions, and so this setting has a game-theoretic interpretation: The stationary solution of the controllers is a Nash equilibrium, which means noncooperative MPC can model competition between two agents/systems. Moreover, closed-loop stability depends on the MPC design in a complex, non-obvious way.

Stackelberg games (von Stackelberg, 1952) represent a different interconnection structure that has *not* been well-studied in the context of distributed MPC. These games have a leader-follower structure, in which the follower's controller is fixed and the leader engineers their own controller. It differs from hierarchical MPC in that only the leader's controller is engineered and the follower's controller may be unstable, and it differs from noncooperative MPC in that the follower gets to first observe the leader's control actions and then choose their own control.

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Though Stackelberg games have been considered in controls (Basar and Selbuz, 1979; Li et al., 2002; Aswani and Tomlin, 2011; Zhu and Martínez, 2011; Krichene et al., 2014), little attention has been paid towards questions of controllability, stability, and synthesis. Stackelberg games are bilevel programs (Colson et al., 2007), which are optimization problems where some constraints are the solutions to a *lower-level* optimization problem. Given recent innovations in the solution of bilevel programs (Aswani et al., 2015; Aswani and Ouattara, 2016), this is a promising framework from which to study control-theoretic questions (like stability and synthesis) of Stackelberg games.

This paper defines and studies bilevel MPC (BiMPC), which is a distributed MPC with Stackelberg game structure. We first formulate BiMPC, and then give counterexamples to demonstrate how interconnections in BiMPC can lead to loss/gain of controllability or stability. Next, we provide sufficient conditions under specific dynamics and cost functions for stabilizability of BiMPC and stability of a given BiMPC controller. An approach to synthesize a stable BiMPC controller is also given. We conclude with one example to demonstrate a duality-based algorithm (Aswani and Ouattara, 2016) for solving the optimization problem for BiMPC, and another example to demonstrate an integer programming algorithm (Aswani et al., 2016a,b) for solving the optimization problem for BiMPC of an electric utility changing electricity prices for demand response of a home's air conditioner controlled by linear MPC.

2. FORMULATION OF BILEVEL MPC

Let $\langle r \rangle = \{0, \dots, r-1\}$. Consider the nonnegative functions $W, Z : \mathbb{R}^{m+p} \rightarrow \mathbb{R}_+$ and $w, z : \mathbb{R}^{m+p+q} \rightarrow \mathbb{R}_+$, and suppose $f : \mathbb{R}^{m+p+q} \rightarrow \mathbb{R}^m$ and $g : \mathbb{R}^{m+p+q} \rightarrow \mathbb{R}^p$. We refer to $x_+ = f(x, y, u, v)$ as the upper dynamics, to $y_+ = g(x, y, u, v)$ as the lower dynamics, and to both dynamics jointly as the overall control system.

We define the lower-level model predictive control (LoMPC) problem with a horizon of N to be

$$\begin{aligned} \mathbf{P}_L(x_0, y_0, u_k) = \min & Z(x_N, y_N) + \sum_{n=0}^{N-1} z(x_n, y_n, u_n, v_n) \\ \text{s.t. } & x_{k+1} = f(x_k, y_k, u_k, v_k) \text{ for } k \in \langle N \rangle \\ & y_{k+1} = g(x_k, y_k, u_k, v_k) \text{ for } k \in \langle N \rangle \\ & y_k \in \mathcal{Y}_\Omega, v_k \in \mathcal{V} \text{ for } k \in \langle N \rangle \\ & y_N \in \mathcal{Y}_\Omega \end{aligned} \quad (1)$$

where \mathcal{Y}_Ω is a positively robust invariant set such that there exists continuous $\ell_L(y)$ and $\bar{\mathcal{X}}, \bar{\mathcal{U}}$ so that

$$\begin{aligned} \mathcal{Y}_\Omega \subseteq \mathcal{Y} \text{ and } \ell_L(y) \in \mathcal{V} \text{ for all } y \in \mathcal{Y}_\Omega \\ g(x, y, u, \ell_L(y)) \in \mathcal{Y}_\Omega \text{ for all } y \in \mathcal{Y}_\Omega, x \in \bar{\mathcal{X}}, u \in \bar{\mathcal{U}} \end{aligned} \quad (2)$$

The set \mathcal{Y}_Ω can be computed for many systems (Mayne and Schroeder, 1997; Kolmanovsky and Gilbert, 1998; Blanchini, 1999; Rakovic et al., 2005).

We define the bilevel model predictive control (BiMPC) problem with a horizon of N to be

$$\begin{aligned} \mathbf{P}_B(x_0, y_0) = \min & W(x_N, y_N) + \sum_{n=0}^{N-1} w(x_n, y_n, u_n, v_n) \\ \text{s.t. } & x_k \in \mathcal{X}_\Omega, u_k \in \mathcal{U} \text{ for } k \in \langle N \rangle \\ & x_N \in \mathcal{X}_\Omega \\ & (x_k, y_k, v_k) \in \arg \min \mathbf{P}_L(x_0, y_0, u_k) \end{aligned} \quad (3)$$

where \mathcal{X}_Ω is a positively robust invariant set such that there exists continuous $\ell_U(x)$ so that

$$\begin{aligned} \mathcal{X}_\Omega \subseteq \bar{\mathcal{X}} \text{ and } \ell_U(x) \in \bar{\mathcal{U}} \text{ for all } x \in \mathcal{X}_\Omega \\ f(x, y, \ell_U(x), v) \in \mathcal{X}_\Omega \text{ for all } x \in \mathcal{X}_\Omega, y \in \mathcal{Y}_\Omega, v \in \mathcal{V} \end{aligned} \quad (4)$$

This set can be computed using the same algorithms cited above. A less conservative choice for the set \mathcal{X}_Ω is

$$\begin{aligned} \mathcal{X}_\Omega \subseteq \bar{\mathcal{X}} \text{ and } \ell_U(x) \in \bar{\mathcal{U}} \text{ for all } x \in \mathcal{X}_\Omega \\ f(x, y, \ell_U(x), v_0) \in \mathcal{X}_\Omega \text{ for all } x \in \mathcal{X}_\Omega, y \in \mathcal{Y}_\Omega, \\ v_0 \in \arg \min \mathbf{P}_L(x, y, \ell_U(X)) \end{aligned} \quad (5)$$

Though this differs from typical invariant sets, this set is computable using standard algorithms for linear dynamics.

We also need to make one mild technical assumption. Let $\mathbf{FP}_L(x_0, y_0)$ be the problem defined as $\mathbf{P}_L(x_1, y_1, u_k)$ with the additional constraints $u_k = \ell_U(x_k)$.

Assumption 1. If $\mathbf{FP}_L(x_0, y_0)$ is feasible, then there is a solution to $\min \mathbf{FP}_L(x_0, y_0)$. Similarly, if $\mathbf{P}_B(x_0, y_0)$ is feasible, then there is a solution to $\min \mathbf{P}_B(x_0, y_0)$.

Our formulation of BiMPC is constructed to ensure recursive feasibility, so that we can focus in the remainder of this paper on questions of controllability, stability, control synthesis, and control engineering applications. We have:

Theorem 2. If there is a feasible solution for $\mathbf{P}_B(x_0, y_0)$, then there is a feasible solution for $\mathbf{P}_B(x_1, y_1)$, where x_1, y_1 are the states after inputs u_0, v_0 feasible for $\mathbf{P}_B(x_0, y_0)$. Moreover, we have $u_0 \in \mathcal{U}$, $v_0 \in \mathcal{V}$, $x_1 \in \mathcal{X}_\Omega$, and $y_1 \in \mathcal{Y}_\Omega$.

Proof. If $\{u_k, v_k \text{ for } k \in \langle N \rangle\}$ is feasible for $\mathbf{P}_B(x_0, y_0)$, then by definition this means the states x_1, y_1 after inputs u_0, v_0 are such that $u_0 \in \mathcal{U}$, $v_0 \in \mathcal{V}$, $x_1 \in \mathcal{X}_\Omega$, and $y_1 \in \mathcal{Y}_\Omega$. Next consider the optimization problem $\mathbf{FP}_L(x_1, y_1)$ defined as $\mathbf{P}_L(x_1, y_1, u_k)$ with the additional constraints $u_k = \ell_U(x_k)$. By the definitions of \mathcal{X}_Ω and \mathcal{Y}_Ω , we have that $v_k = \ell_L(y_k)$ is feasible for $\mathbf{FP}_L(x_1, y_1)$. So by Assumption 1 this means there exists optimal $v_k^* \in \arg \min \mathbf{FP}_L(x_1, y_1)$. And so the point corresponding to the overall control system with initial condition x_1, y_1 and inputs $u_k = \ell_U(x_k)$ and v_k^* is feasible for $\mathbf{P}_B(x_1, y_1)$. \square

3. INTERCONNECTION COUNTEREXAMPLES

The interconnection in BiMPC between the upper and lower dynamics leads to behavior often not seen in hierarchical or noncooperative control. We show the interconnection can cause a loss/gain of controllability or stability.

3.1 Controllability Examples

Our first example is a problem where the LoMPC is

$$\begin{aligned} \mathbf{P}_L(x_0, y_0, u_k) = \min & (y_1)^2 \\ \text{s.t. } & x_1 = 2x_0 + y_0 \\ & y_1 = y_0 + u_0 + v_0 \\ & y_1 \in \mathbb{R}, v_0 \in \mathbb{R} \end{aligned} \quad (6)$$

The overall control system is controllable in u . But a simple calculation shows the control of LoMPC is $v = -y - u$, and so for $n \geq 1$ the dynamics seen by BiMPC are

$$\begin{aligned} x_{n+1} &= 2x_n \\ y_{n+1} &= 0 \end{aligned} \quad (7)$$

which is not controllable in u . This example is interesting because the control action of LoMPC leads to a loss of controllability by BiMPC.

The next example is a problem where the LoMPC is

$$\begin{aligned} \mathbf{P}_L(x_0, y_0, u_k) = \min & (y_1)^2 + (v_0)^2 \\ \text{s.t. } & x_1 = 2x_0 + v_0 \\ & y_1 = y_0 + u_0 + v_0 \\ & y_1 \in \mathbb{R}, v_0 \in \mathbb{R} \end{aligned} \quad (8)$$

The overall control system is not controllable in u . But a simple calculation shows that the control of LoMPC is $v = -(y + u)/2$, and so the dynamics seen by BiMPC are

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} 4 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \frac{1}{2} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} u_n \quad (9)$$

which is controllable in u . This example is interesting because the control action of LoMPC leads to a gain of controllability by BiMPC.

3.2 Stability Examples

Our first example is a problem where the LoMPC is

$$\begin{aligned} \mathbf{P}_L(x_0, y_0, u_k) = \min & (y_1)^2 + (v_0)^2 \\ \text{s.t. } & x_1 = x_0 + 2y_0 + u_0 \\ & y_1 = 2x_0 + y_0 + u_0 + v_0 \\ & y_1 \in \mathbb{R}, v_0 \in \mathbb{R} \end{aligned} \quad (10)$$

and the BiMPC is

$$\begin{aligned} \mathbf{P}_B(x_0, y_0) = \min & (x_1)^2 + (u_0)^2 \\ \text{s.t. } & x_1 \in \mathbb{R}, u_0 \in \mathbb{R} \\ & (x_1, y_1, v_0) \in \arg \min \mathbf{P}_L(x_0, y_0, u_k) \end{aligned} \quad (11)$$

A simple calculation gives that the closed loop system is

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \frac{1}{4} \cdot \begin{bmatrix} 2 & 4 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} \quad (12)$$

which is unstable. This is interesting because the lower dynamics are stable when $(x, u) = (0, 0)$, and the upper dynamics are stable when $(y, v) = (0, 0)$; yet the overall control system is unstable. However, by changing the objective function of the BiMPC to $\min(x_1)^2 + (u_0)^2 + (y_1)^2$, the closed loop system becomes

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \frac{1}{5} \cdot \begin{bmatrix} 1 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} \quad (13)$$

which is stable. Thus stability of the overall control system is dependent on the gains of the upper and lower dynamics.

As another example, consider the LoMPC given by

$$\begin{aligned} \mathbf{P}_L(x_0, y_0, u_k) = \min & (v_0)^2 \\ \text{s.t. } & x_1 = x_0 + 4y_0 + u_0 \\ & y_1 = 4y_0 + u_0 + v_0 \\ & y_1 \in \mathbb{R}, v_0 \in \mathbb{R} \end{aligned} \quad (14)$$

and the BiMPC is

$$\begin{aligned} \mathbf{P}_B(x_0, y_0) = \min & (x_1)^2 + (u_0)^2 \\ \text{s.t. } & x_1 \in \mathbb{R}, u_0 \in \mathbb{R} \\ & (x_1, y_1, v_0) \in \arg \min \mathbf{P}_L(x_0, y_0, u_k) \end{aligned} \quad (15)$$

A simple calculation gives that the closed loop system is

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} 1 & 4 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} \quad (16)$$

which is unstable. This example is interesting because the control provided by LoMPC (the control is always $v \equiv 0$) is never stabilizing, while the control of BiMPC stabilizes

the upper dynamics when $(y, v) = (0, 0)$. On the other hand, when the objective function of BiMPC is changed to $\min(x_1)^2 + (u_0)^2 + (y_1)^2$, the closed loop system is

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \frac{1}{5} \cdot \begin{bmatrix} 4 & 4 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} \quad (17)$$

which is stable. This example shows that in certain situations the BiMPC can stabilize the overall control system independent of the control action provided by LoMPC.

For our final example, consider the LoMPC given by

$$\begin{aligned} \mathbf{P}_L(x_0, y_0, u_k) = \min & (v_0)^2 \\ \text{s.t. } & x_1 = x_0 + 2y_0 + u_0 \\ & y_1 = -\frac{1}{2}y_0 + u_0 + v_0 \\ & y_1 \in \mathbb{R}, v_0 \in \mathbb{R} \end{aligned} \quad (18)$$

and the BiMPC is

$$\begin{aligned} \mathbf{P}_B(x_0, y_0) = \min & (x_1)^2 + (u_0)^2 \\ \text{s.t. } & x_1 \in \mathbb{R}, u_0 \in \mathbb{R} \\ & (x_1, y_1, v_0) \in \arg \min \mathbf{P}_L(x_0, y_0, u_k) \end{aligned} \quad (19)$$

A simple calculation gives that the closed loop system is

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \frac{1}{2} \cdot \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} \quad (20)$$

which is unstable. This example is interesting because the control provided by LoMPC (the control is always $v \equiv 0$) leads to stable lower dynamics when $(x, u) = (0, 0)$; however, the control action of BiMPC destabilizes the lower dynamics while stabilizing the upper dynamics. But when the objective function of BiMPC is changed to $\min(x_1)^2 + (u_0)^2 + 2(y_1)^2$, the closed loop system is

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \frac{1}{4} \cdot \begin{bmatrix} 3 & 7 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} \quad (21)$$

which is stable. This example shows the BiMPC must be carefully designed to ensure closed loop stability.

4. SUFFICIENT CONDITION FOR STABILITY

The above examples show that stability of BiMPC depends non-trivially on the dynamics and cost functions, and so we focus on providing one set of sufficient conditions for stability when the objectives are positive semidefinite quadratic functions and the dynamics are linear:

Let $\xi' = [x' \ y']$ and $\nu' = [u' \ v']$, and suppose the overall control system is linear $\xi_+ = A\xi + B\nu$. The upper and lower dynamics are $x_+ = A_{11}x + A_{12}y + B_{11}u + B_{12}v$ and $y_+ = A_{21}x + A_{22}y + B_{21}u + B_{22}v$, respectively, where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}. \quad (22)$$

Furthermore, we assume the objectives are quadratic positive semidefinite functions $W(\xi_N) = \xi_N' P \xi_N$, $w(\xi_n, \nu_n) = \xi_n' Q \xi_n + \nu_n' R \nu_n$, $Z(\xi_N) = y_N' S y_N$, and $z(\xi_n, \nu_n) = y_n' T y_n + \nu_n' U \nu_n$, where P, Q, R, S, T, U are positive semidefinite matrices with the block structure

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \quad (23)$$

where $R'_{11} = R_{11}$, $R'_{12} = R_{21}$, and $R'_{22} = R_{22}$.

We begin with two propositions that will be used in our proof for a set of sufficient conditions for stability.

Proposition 3. The solution to the finite-horizon discrete-time LQR problem with fixed u_0

$$\begin{aligned} \min & y_1' S y_1 + y_0' T y_0 + v_0' U v_0 \\ \text{s.t. } & \xi_1 = A \xi_0 + B v_0 \end{aligned} \quad (24)$$

is (when U is invertible) given by $v_0 = E \xi_0 + F u_0$, where we have that $E = -(B_{22}' S B_{22} + U)^{-1} [A_{21} \ A_{22}]$ and $F = -(B_{22}' S B_{22} + U)^{-1} B_{21}$.

Proof. This follows from a simple dynamic programming (DP) calculation. \square

Proposition 4. Consider the finite-horizon discrete-time LQR problem

$$\begin{aligned} \min & \xi_1' P \xi_1 + \xi_0' \bar{Q} \xi_0 + u_0' \bar{R} u_0 + 2 \xi_0' \bar{\Gamma} u_0 \\ \text{s.t. } & \xi_1 = \bar{A} \xi_0 + \bar{B} u_0 \end{aligned} \quad (25)$$

where $\bar{Q} = Q + E' R_{22} E$, $\bar{R} = R_{11} + R_{12} F + F' R_{12}' + F' R_{22} F$, $\bar{\Gamma} = E' R_{12}' + E' R_{22} F$,

$$\bar{A} = A + \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix} E \quad \text{and} \quad \bar{B} = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} + \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix} F. \quad (26)$$

The solution is (when R is invertible) given by $u_0 = K \xi_0$, where $K = -(\bar{B}' P \bar{B} + \bar{R})^{-1} (\bar{B}' P \bar{A} + \bar{\Gamma}')$.

Proof. This follows from a simple DP calculation. \square

Having shown the above two propositions, we can now provide one set of sufficient conditions for stability:

Theorem 5. If $\bar{A} + \bar{B} K$ is Schur stable, then the BiMPC control with $N = 1$ is locally asymptotically stable (LAS).

Proof. For all x_0, y_0 small enough the solution to LoMPC matches the LQR solution in Proposition 3, and thus for these x_0, y_0 the solution to BiMPC matches the LQR solution in Proposition 4. So for all x_0, y_0 small enough the overall control system with control generated by BiMPC generates the closed-loop system $\xi_+ = (\bar{A} + \bar{B} K) \xi$, which is Schur stable. This implies that the overall control system is stable for all x_0, y_0 small enough. \square

A related question is how to synthesize a stable BiMPC controller. The following describes one such procedure.

Theorem 6. Suppose (\bar{A}, \bar{B}) is stabilizable, and choose any $Q, R > 0$. If we choose any H such that $\bar{A} + \bar{B} H$ is Schur Stable, then for the P that solves the Lyapunov equation

$$\begin{aligned} (\bar{A} + \bar{B} H)' P (\bar{A} + \bar{B} H) - P \\ = -(\bar{Q} + H' \bar{R} H + H' \bar{\Gamma}' + \bar{\Gamma} H) \end{aligned} \quad (27)$$

the closed-loop system under BiMPC with $N = 1$ is LAS.

Proof. Existence of H is a well-known property of stabilizability (Callier and Desoer, 1994), and the Lyapunov equation (27) has a unique positive definite solution because $\bar{Q} + H' \bar{R} H + H' \bar{\Gamma}' + \bar{\Gamma} H$ is positive definite and $A + B H$ is Schur stable (Lewis and Syrmos, 1995). Let V_0 be the minimum of (25) with initial condition ξ_0 , and let V_1 be the minimum of (25) with initial condition ξ_1 . If $u_1 = H x_1$, then $V_1 - V_0 \leq \xi_1' X \xi_1 - \xi_0' \bar{Q} \xi_0 - u_0' \bar{R} u_0 - 2 \xi_0' \bar{\Gamma} u_0$ where $X = (\bar{A} + \bar{B} H)' P (\bar{A} + \bar{B} H) - P + \bar{Q} + H' \bar{R} H + H' \bar{\Gamma}' + \bar{\Gamma} H$. Note $X = 0$ by (27), and so $V_1 - V_0 \leq -\xi_0' \bar{Q} \xi_0 - u_0' \bar{R} u_0 - 2 \xi_0' \bar{\Gamma} u_0$. But we can rewrite this right hand side as

$$-\xi_0' Q \xi_0 - u_0' R_{11} u_0 - \begin{bmatrix} \xi_0 \\ u_0 \end{bmatrix}' \begin{bmatrix} 0 & 0 \\ E & F \end{bmatrix}' R \begin{bmatrix} 0 & 0 \\ E & F \end{bmatrix} \begin{bmatrix} \xi_0 \\ u_0 \end{bmatrix} \quad (28)$$

which is negative definite in (ξ_0, u_0) . A similar argument shows the value function of (25) is positive definite in (ξ_0, u_0) . So this value function is a Lyapunov function for the closed-loop system with $u = K \xi$, implying $\bar{A} + \bar{B} K$ is Schur stable. The result now follows from Theorem 5. \square

The above says that if (\bar{A}, \bar{B}) is stabilizable then there exists P, Q, R that ensure the BiMPC is LAS. This gives the following sufficient condition for stabilizability of BiMPC.

Corollary 7. If (\bar{A}, \bar{B}) is stabilizable, then BiMPC is also stabilizable.

5. ALGORITHMS FOR SOLVING BIMPC

This section describes two numerical approaches to solving the optimization problem defining BiMPC, and the algorithms were implemented in MATLAB 2015b on a laptop computer with a 2.4GHZ processor and 16GB RAM.

5.1 Duality Approach to Solving BiMPC

New algorithms that use duality theory to solve bilevel programs have recently been proposed (Aswani et al., 2015; Aswani and Ouattara, 2016), and here we use one such algorithm to solve BiMPC.

Suppose the LoMPC is

$$\begin{aligned} \mathbf{P}_L(x_0, y_0, u_k) = \min & y_1^2 + v_0^2 \\ \text{s.t. } & x_1 = 2x_0 + y_0 + u_0 \\ & y_1 = 2y_0 + u_0 + v_0 \\ & v_0 \in [-3, 3], y_1 \in [-1, 1] \end{aligned} \quad (29)$$

and the BiMPC is

$$\begin{aligned} \mathbf{P}_B(x_0, y_0) = \min & p_{11} x_1^2 + 3y_1^2 + x_0^2 + y_0^2 + u_0^2 + v_0^2 \\ \text{s.t. } & (y_1, v_0) \in \arg \min \mathbf{P}_L(x_0, y_0, u_k) \\ & u_0 \in [-2, 2], x_1 \in [-1, 1] \end{aligned} \quad (30)$$

where $p_{11} = 7.274$. A simple calculation shows that (\bar{A}, \bar{B}) is stabilizable and $\bar{A} + \bar{B} K$ is Schur stable. Hence by Theorem 5, this BiMPC controller is LAS. We can numerically solve this BiMPC using the approach in (Aswani and Ouattara, 2016), which reformulates $\mathbf{P}_B(x_0, y_0)$ as

$$\begin{aligned} \mathbf{DP}(x_0, y_0) = \min & p_{11} x_1^2 + 3y_1^2 + x_0^2 + y_0^2 + u_0^2 + v_0^2 \\ \text{s.t. } & x_1 = 2x_0 + y_0 + u_0 \\ & y_1 = 2y_0 + u_0 + v_0 \\ & y_1^2 + v_0^2 - h_\mu(\lambda, \eta, x_0, y_0, u_0) \leq \epsilon \\ & u_0 \in [-2, 2], v_0 \in [-3, 3] \\ & x_1, y_1 \in [-1, 1], \lambda \geq 0 \end{aligned} \quad (31)$$

where $\epsilon, \mu \geq 0$ are regularization terms, and $h_\mu = \min_{\zeta} \{ \mu \|\zeta\|^2 + \zeta_1^2 + \zeta_2^2 + \eta_1(2y_0 + u_0 + \zeta_1 - \zeta_2) + \eta_2(2x_0 + y_0 + u_0 - \zeta_3) + \lambda_1(1 - \zeta_2) + \lambda_2(\zeta_2 - 1) + \lambda_3(3 - \zeta_1) + \lambda_4(\zeta_1 - 3) | \zeta_i \in [-4, 4] \}$ is a Regularized Lagrangian Dual (RDF) that is constructed to be differentiable (Aswani and Ouattara, 2016). (The usual Lagrangian Dual is only directionally differentiable.) There is also an equation for the gradient of the RDF, and so we can use standard nonlinear optimization software to solve $\mathbf{DP}(x_0, y_0)$. Simulation results with $\mu = \epsilon = 0.01$ are shown in Fig. 1.

5.2 Demand Response for Home Air-Conditioner

Electric utilities use demand response (DR) to better match the usage and generation of electricity, and one

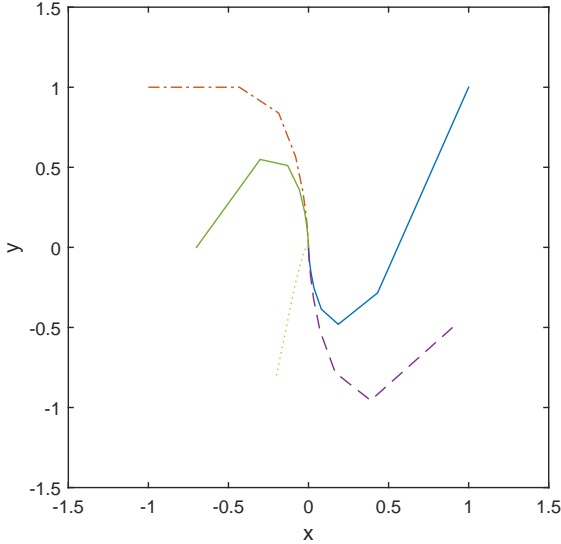


Fig. 1. A phase plot (with 5 initial conditions) for the overall system with BiMPC in (30) solved using a duality approach (Aswani and Ouattara, 2016).

approach is time-of-day pricing to disincentivize electricity usage during peak demand hours. But there is a question of how the utility should choose the prices. Here, we use BiMPC to choose pricing for a home with an air-conditioner controlled by linear MPC. This scenario is motivated by recent work on using MPC to control HVAC (Deng et al., 2010; Ma et al., 2011; Aswani et al., 2012a,b,c; He and Gonzalez, 2016), and is similar to the bilevel approach described in (Zugno et al., 2013).

In particular, consider a single home that uses the following (simplified) linear MPC to control an air-conditioner:

$$\begin{aligned} \mathbf{P}_L(T_0, c_k) = \min \quad & \sum_{n=0}^N (T_n - T_d)^2 + \sum_{n=0}^N r \cdot c_n u_n \\ \text{s.t.} \quad & T_{k+1} = K_r T_k - K_c u_k + K_w w_k + q \\ & T_k \in [20, 24], u_k \in [0, 0.5] \end{aligned} \quad (32)$$

where T_n is room temperature ($^{\circ}\text{C}$), T_d is desired room temperature, r quantifies the home owner's trade off between comfort and cost, c_n is electricity price (cents/kWh), u_n is the duty cycle of the air-conditioner, w_n is outdoor temperature, and q is heating due to occupancy. The sampling period is 15 minutes, and the parameter values $K_r = 0.64$, $K_c = 2.64$, $K_w = 0.10$, $q = 6.98$ are from the HVAC model in (Aswani et al., 2012a).

If the electric utility would like to reduce electricity consumption during 1PM-5PM, then the problem of choosing time-of-day pricing can be written as the BiMPC given by

$$\begin{aligned} \mathbf{P}_B(T_0) = \min \quad & \sum_{n \in H_p} u_n \\ \text{s.t.} \quad & (T_k, u_k) \in \arg \min \mathbf{P}_L(T_0, c_k) \\ & c_k \in [5, 10] \end{aligned} \quad (33)$$

where H_p are the indices that correspond to 1PM-5PM. This BiMPC can be solved using the integer programming reformulation described in (Aswani et al., 2016a,b), which rewrites $(T_k, u_k) \in \arg \min \mathbf{P}_L(T_0, c_k)$ as KKT conditions (Boyd and Vandenberghe, 2004) and then uses binary variables to exactly linearize the reformulation. For $\mathbf{P}_B(T_0)$, the resulting mixed-integer linear program (MILP) is

$$\begin{aligned} \mathbf{P}_B(T_0) = \min \quad & \sum_{n \in H_p} u_n \\ \text{s.t.} \quad & T_{k+1} = K_r T_k + K_c u_k + K_w w_k + q \\ & 2(T_k - T_d) + \mu_{k-1} - K_r \mu_k = \lambda_k^2 - \lambda_k^1 \\ & r c_k - K_c \mu_k + \lambda_k^3 - \lambda_k^4 = 0 \\ & 0 \leq \lambda_k^1 \leq M y_k^1, \quad 0 \leq \lambda_k^2 \leq M x_k^1 \\ & 24 y_k^1 + 20(1 - y_k^1) \leq T_k \\ & T_k \leq 20 x_k^1 + 24(1 - x_k^1) \\ & 0 \leq \lambda_k^3 \leq M y_k^2, \quad 0 \leq \lambda_k^4 \leq M x_k^2 \\ & 0.5 y_k^2 \leq u_k \leq x_k^2 + 0.5(1 - x_k^2) \\ & c_k \in [5, 10], \quad x_k, y_k \in \{0, 1\} \end{aligned} \quad (34)$$

where μ_n, λ_n^i are Lagrange multipliers for equality (i.e., dynamics of T_k) and inequality constraints (i.e., upper/lower bounds on T_k and u_k) in $\mathbf{P}_L(T_0, c_k)$, and x_n^j, y_n^j are binary variables that linearize complimentary slackness.

This MILP can be solved by standard software, and we used the Gurobi solver (Gurobi Optimization, 2016) and CVX toolbox (Grant and Boyd, 2014). Simulation results with $T_0 = 23$, $T_d = 22$, $N = 96$ (1 day), and $w_n = 17.5 - 2.5 \cos(\pi n/48)$ are shown in Fig. 2, and the chosen price induces the HVAC controller to precool the room to reduce electricity consumption between 1PM-5PM (which was the DR goal of the electric utility). Although solving a MILP is NP-hard, the solution time was on average 0.35s, with a minimum of 0.19s and maximum of 1.68s.

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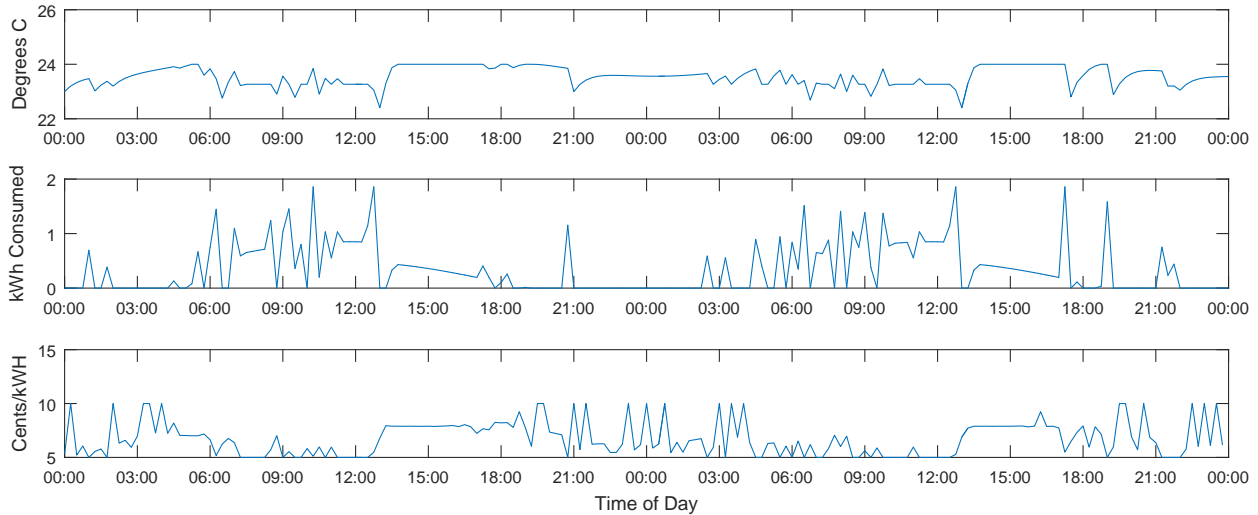


Fig. 2. A simulation of the BiMPC controller shows reduced electricity consumption between 1PM-5PM by choosing electricity time-of-day pricing that induces the linear MPC of the HVAC to precool the room in the morning. From top to bottom the plots are room temperature, the amount of HVAC power consumption, and the electricity price.

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